Kaluza-Klein Method in Theory of Rotating Quantum Fields

D.V. Fursaev*[1]

[1] *Joint Institute for Nuclear Research, Bogoliubov Laboratory of Theoretical Physics, 141 980 Dubna, Russia*

Abstract

Quantum fields on a stationary space-time in a rotating Killing reference frame are considered. Finding solutions of wave equations in this frame is reduced to a fiducial problem on a static background. The rotation results in a gauge connection in a way similar to appearance of gauge fields in Kaluza-Klein models. Such a Kaluza-Klein method in theory of rotating quantum fields enables one to simplify computations and get a number of new results similar to those established for static backgrounds. In particular, we find with its help functional form of free energy at high temperatures. Applications of these results to quantum fields near rotating black holes are briefly discussed.

PACS number(s): O4.62.+v, 03.65.Pm, 04.70.Dy

*e-mail: fursaev@thsun1.jinr.ru

1 Introduction

It is very well known that computations of quantum effects are simplified on static space-times where some general results such as high-temperature asymptotics [1]–[3] can be established. Sometimes explicit computations can be done in case of rigidly rotating fields if the background is static and axially-symmetric. Recent results of this kind are high-temperature asymptotics of rotating CFT's on Einstein manifolds studied in [4]–[7]. Other similar computations can be found in [8]. However, methods used for rotating fields on static space-times cannot be applied to other important situations, to the Kerr geometry, for example. The aim of this paper is to suggest an approach how to deal with quantum effects on stationary geometries and to get with its help some new results.

The basic idea of our approach was formulated in [9] and it is to reduce the problem on a stationary background $\mathcal{M}$ to an equivalent but more simple problem on a fiducial static space-time $\tilde{\mathcal{M}}$. This can be done as follows. Consider a Killing frame of reference on $\mathcal{M}$, i.e., a frame related to observers with velocities u^μ parallel to the time-like Killing field ξ^μ. The space-time metric in this frame can be represented as $ds^2 = dl^2 - B(u_\mu dx^\mu)^2$, where $B = -\xi^2$ and $dl^2 = h_{\mu\nu}dx^\mu dx^\nu$ is the proper distance between the points x^μ and $x^\mu + dx^\mu$. It can be shown that wave equations of fields on $\mathcal{M}$ are reduced to equations on a fiducial static space-time $\tilde{\mathcal{M}}$ with metric $d\tilde{s}^2 = dl^2 - Bdt^2$. As a result of rotation of the frame the covariant derivative on $\tilde{\mathcal{M}}$ takes the form $\tilde{\nabla}_\mu - a_\mu \partial_t$, where $a_\mu dx^\mu = -u_i dx^i/\sqrt{B}$. For a solution $\phi_\omega(t, x^i) = e^{-i\omega t}\phi_\omega(x^i)$ with a certain frequency ω the vector a_μ is just a gauge potential while ω is a charge. Appearance of a_μ in our case is analogous to appearance of the gauge potential in Kaluza-Klein theories and for this reason we call this method the Kaluza-Klein (KK) method. One can consider now a related static problem for a fiducial charged field $\phi^{(\lambda)}$ living on $\tilde{\mathcal{M}}$ and interacting with the gauge field a_μ. The charge of the field is λ, a real parameter. If solutions $\phi^{(\lambda)}$ of field equations on $\tilde{\mathcal{M}}$ are known for different λ one can identify the single-particle excitation of the physical field carrying energy ω with the fiducial field having the same energy and charge $\lambda = \omega$, i.e., to put $\phi_\omega = \phi_\omega^{(\omega)}$.

In some cases the fiducial problem can be used and has certain advantages. For instance, for fields near a rotating black hole the spectrum of ω is continuous and is specified by the density of levels $dn/d\omega$. In this and other similar problems $dn/d\omega$ plays an important role in computing physical quantities, however, direct derivation of $dn/d\omega$ by using eigen modes is quite complicated. On the other hand, for fiducial fields there are methods developed for static backgrounds which do not require knowing eigen modes explicitly. In this paper we establish relation between $dn/d\omega$ and the heat kernel of the operator $H^2(\lambda)$, where $H(\lambda)$ is the one-particle Hamiltonian of $\phi^{(\lambda)}$. We, thus, find a way how to compute $dn/d\omega$ by using powerful heat kernel techniques. This enables us to get a number of new results. In particular, we compute the free of a rotating quantum field

at high temperatures

$$F(T) = -\int d^3x\sqrt{-g}\left[aT^4 + T^2(\Phi + b\Omega^2) + O(\ln T)\right] \quad . \tag{1.1}$$

Here a, b are numerical coefficients determined by the spin of fields, T is the local Tolman temperature measured by the Killing observer. The term $\Phi = \Phi(m, R, w)$ depends on the mass m of the field, scalar curvature R and acceleration w^μ of the frame. This term and the coefficient a are the same as for static spaces [2], [3]. The rotation results in the new term which depends on the angular velocity Ω of the Killing frame measured with respect to a local Lorentz frame. In principle, our method also enables one to find explicitly other terms in temperature expansion (1.1) and, in particular, $\ln T$ term in (1.1) which we present later in the text.

The rest of the paper is organized as follows. The Kaluza-Klein method is discussed in Section 2. We introduce the Killing frame of reference, and show how to formulate the fiducial problem for scalar and spinor fields. The systems with continuous spectrum are discussed in Section 3. We introduce the density of levels $dn/d\omega$ and find out its relation to the heat kernel of $H^2(\lambda)$. Applications of our method are discussed in the second part of the paper. By using the heat kernel technique we obtain high-frequency asymptotics of $dn/d\omega$ (Section 4) and with its help high-temperature behaviour of the free energy (Section 5). In Section 5 we also briefly discuss quantum theory near rotating black holes. Further comments and a discussion of our results can be found in Section 6. Some geometrical relations in the Killing frame are presented in Appendix A. Appendix B clarifies some technical issues which appear in Sections 3.

2 The method

2.1 Killing reference frame

Let us consider a field ϕ on a domain $\mathcal{M}$ of a D-dimensional space-time where there is a time-like Killing vector field ξ^μ ($\xi^2 < 0$). $\mathcal{M}$ may be a complete manifold if ξ^μ is everywhere time-like. In most other cases ξ^μ is time-like only in some region. We will study solutions of field equations in the frame of reference related to Killing observers whose velocity u^μ is parallel to ξ^μ

$$u^\mu = B^{-1/2}\xi^\mu \quad , \quad B = -\xi^2 \quad . \tag{2.1}$$

For a Killing observer a solution ϕ_ω carrying the energy ω is defined as

$$i\mathcal{L}_\xi\phi_\omega = \omega\phi_\omega \tag{2.2}$$

where $\mathcal{L}_\xi$ is the Lie derivative along ξ^μ. The background metric $g_{\mu\nu}$ can be represented as

$$g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu \quad , \tag{2.3}$$

where $h_{\mu\nu}$ is the projector on the directions orthogonal to u_μ. Because sheer and expansion of the family of Killing trajectories vanish identically the trajectories are characterized in each point only by their acceleration w_μ and the rotation $A_{\mu\nu}$ with respect to a local Lorentz frame [10]

$$w_\mu = u_{\mu;\lambda}u^\lambda \quad , \tag{2.4}$$

$$A_{\mu\nu} = \frac{1}{2}h_\mu^\lambda h_\nu^\rho(u_{\lambda;\rho} - u_{\rho;\lambda}) \quad . \tag{2.5}$$

To proceed it is convenient to choose coordinates $x^\mu = (t, x^i)$, $i = 1, D-1$, where $\xi = \partial/\partial t$ and, consequently, $h_{0\mu} = 0$. According with (2.1), (2.3), the interval on $\mathcal{M}$ can be written as

$$ds^2 = -Bd\tau^2 + dl^2 \quad , \tag{2.6}$$

$$d\tau = -\frac{1}{\sqrt{B}}(u_\mu dx^\mu) = dt + a_i dx^i \quad , \quad a_i = -\frac{u_i}{\sqrt{B}} \tag{2.7}$$

$$dl^2 = h_{\mu\nu}dx^\mu dx^\nu = h_{ij}dx^i dx^j \quad . \tag{2.8}$$

The vector a_i can be used to synchronize the clocks in points with coordinates x^i and $x^i + dx^i$. The metric dl^2 serves to measure the proper distance between these points. In the coordinates (t, x^i) the only non-zero components of acceleration (2.4) and rotation (2.5) are

$$w_i = \frac{1}{2}(\ln B)_{,i} \quad , \quad A_{ij} = -\frac{1}{2}\sqrt{B}(a_{i,j} - a_{j,i}) \quad . \tag{2.9}$$

In four-dimensional space-time one can define a vector of local angular velocity

$$\Omega_i = \frac{1}{2}\epsilon_{ijk}A^{jk} \quad , \tag{2.10}$$

where ϵ_{ijk} is a totally antisymmetric tensor. The absolute value of the angular velocity is

$$\Omega = (\Omega_i\Omega^i)^{1/2} = \left(\frac{1}{2}A^{\mu\nu}A_{\mu\nu}\right)^{1/2} \quad . \tag{2.11}$$

The form of the metric in the Killing frame, equations (2.6), (2.7), is preserved under arbitrary change of coordinates x^i provided h_{ij} and a_i transform as a $D-1$ dimensional tensor and vector. There is also another group of transformations, which preserves (2.6), (2.7), namely, $t = t' + f(x)$, $a_i = a'_i - \partial_i f(x)$, where f is an arbitrary function of x^i. Under these transformations a_i changes as an Abelian gauge vector field. By considering single-particle excitations with the fixed energy ω

$$\phi_\omega(t, x^i) = e^{-i\omega t}\phi_\omega(x^i) \quad , \tag{2.12}$$

one can realize this group of transformations as a local $U(1)$,

$$\phi_\omega(t, x^i) = \phi'_\omega(t', x^i) = e^{-i\omega t'}\phi'_\omega(x^i) \quad ,$$

$$\phi_\omega(x^i) = e^{i\omega f(x)}\phi'_\omega(x^i) \quad . \tag{2.13}$$

In this picture, ω coincides with an "elementary charge". To quantize in the Killing frame one needs a full set of modes $\phi_\omega(x)$. As follows from the above arguments, the equations which determine $\phi_\omega(x)$ have a form of $D-1$ dimensional equations for charged fields in external gauge field a_i on a space with the metric h_{ij}. It is important that covariant properties of the theory in D dimensions guarantee diffeo and gauge-covariant form of the $D-1$ dimensional problem. Such a reduction from D to $D-1$ is analogous to the Kaluza-Klein procedure which yields the Einstein-Maxwell theory from higher dimensional gravity. The difference between the two reductions is that in the standard Kaluza-Klein approach the "extra" dimensions are compact and the charges are quantized.

Let $\mathcal{B}$ denote a space with metric dl^2, see (2.8). If the Killing trajectories do not rotate, $a_i = 0$, $\mathcal{B}$ can be embedded in $\mathcal{M}$ as a constant-time hypersurface with the unit normal vector u^μ. If Ω is not vanishing $\mathcal{B}$ cannot be embedded in $\mathcal{M}$ because u_μ cannot be a gradient. Consider now a point p on $\mathcal{B}$ with coordinates x^i and a vector V_i from the tangent space at p. On $\mathcal{M}$, p corresponds to a trajectory of a Killing observer with the same coordinates x^i. At any point of the trajectory one can define a vector V_μ orthogonal to u^μ such as $V_i = h_i^\mu V_\mu$. Suppose that connection $\tilde{\nabla}_i$ on $\mathcal{B}$ is determined by h_{ij}. Then the covariant derivative with respect to this connection can be written as [10]

$$\tilde{\nabla}_j V_i = h_i^\lambda h_j^\rho V_{\lambda;\rho} \quad , \tag{2.14}$$

where $V_{\mu;\nu}$ is the covariant derivative on $\mathcal{M}$ with respect to the connection defined by $g_{\mu\nu}$. (One can easily check that $\tilde{\nabla}_k h_{ij} = 0$.) Relation (2.14) can be generalized to an arbitrary field on $\mathcal{M}$. For instance, for a scalar field

$$h_j^\mu \partial_\mu \phi = (\partial_j - a_j \partial_t)\phi \equiv D_i \phi \quad , \tag{2.15}$$

for a vector orthogonal to u^μ

$$h_i^\lambda h_j^\rho V_{\lambda;\rho} = (\tilde{\nabla}_j - a_j \partial_t) V_i \equiv D_j V_i \quad , \tag{2.16}$$

where $V_i = h_i^\mu V_\mu$. The time derivative in (2.15), (2.16) appears in general because fields on $\mathcal{M}$ change along the Killing trajectory. If ϕ and V_μ are solutions with certain frequency, see (2.12), then D_i become covariant derivatives on $\mathcal{B}$ in external gauge field a_i. This demonstrates explicitly diffeo and gauge-covariance of the theory which are left after the reduction.

2.2 Scalar fields

To illustrate the KK method we consider first a real scalar field ϕ which satisfies the equation

$$(-\nabla^\mu \nabla_\mu + V)\phi = 0 \quad , \tag{2.17}$$

where V is a potential. In the Killing frame (2.3) the wave operator can be represented as

$$\nabla^\mu\nabla_\mu = -\frac{1}{B}(\xi^\mu\nabla_\mu)^2 + \frac{1}{2B}B^{,\nu}\nabla_\nu + h^{\mu\nu}\nabla_\mu\nabla_\nu = -\frac{1}{B}\partial_t^2 - \frac{1}{2B}B_{,i}h^{ij}D_j + h^{ij}D_iD_j \quad , \tag{2.18}$$

where D_i is defined by (2.15), (2.16). It is easy to see that (2.18) can be written in a D–dimensional form

$$\nabla^\mu\nabla_\mu = \tilde{g}^{\mu\nu}D_\mu D_\nu \quad , \tag{2.19}$$

$$D_\mu = \tilde{\nabla}_\mu - a_\mu\partial_t \quad , \tag{2.20}$$

where $a_\mu dx^\mu = a_i dx^i$. The connections $\tilde{\nabla}_\mu$ are determined on some space $\tilde{\mathcal{M}}$ with the metric

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = -Bdt^2 + dl^2 \quad . \tag{2.21}$$

Relation between $\tilde{\mathcal{M}}$ and $\mathcal{M}$ becomes transparent when comparing (2.21) with (2.6). We will call $\tilde{\mathcal{M}}$ and a_μ the fiducial space-time and the fiducial gauge potential, respectively. Let us consider now a scalar field $\phi^{(\lambda)}$ on $\tilde{\mathcal{M}}$ which obeys the equation

$$(-\tilde{g}^{\mu\nu}(\tilde{\nabla}_\mu + i\lambda a_\mu)(\tilde{\nabla}_\nu + i\lambda a_\nu) + V)\phi^{(\lambda)} = 0 \quad . \tag{2.22}$$

If $\phi^{(\lambda)}_\omega(t,x^i) = e^{-i\omega t}\phi^{(\lambda)}_\omega(x^i)$ is a solution to (2.22) then, as follows from (2.19)–(2.22) $\phi^{(\omega)}_\omega(t,x^i)$ is a solution to (2.17). Therefore, for a scalar filed the relativistic eigen-energy problem in the stationary space-time can be reduced to an analogous problem on a fiducial static background $\tilde{\mathcal{M}}$. Equation (2.22) can be further rewritten in the form

$$H^2(\lambda)\phi^{(\lambda)}_\omega(x^i) = \omega^2\phi^{(\lambda)}_\omega(x^i) \quad , \tag{2.23}$$

where $H(\lambda)$ has the meaning of a relativistic single-particle Hamiltonian for the field $\phi^{(\lambda)}_\omega(x^i)$ on $\mathcal{B}$. For definition and discussion of single-particle Hamiltonians see [11].

2.3 Spinor fields

In the same way one can treat a free spin 1/2 field ψ described by the Dirac equation

$$(\gamma^\mu\nabla_\mu + m)\psi = 0 \quad . \tag{2.24}$$

To this aim one has to represent the Dirac operator as

$$\gamma^\mu\nabla_\mu = \tilde{\gamma}^t\xi^\mu\nabla_\mu + \tilde{\gamma}^i D_i \quad , \tag{2.25}$$

where γ_μ and $\tilde{\gamma}_\mu$ are two sets of gamma-matrices on $\mathcal{M}$ and $\tilde{\mathcal{M}}$, respectively,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \quad , \quad \{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\tilde{g}_{\mu\nu} \quad . \tag{2.26}$$

$$\tilde{\gamma}_t = \xi^\mu\gamma_\mu \quad , \quad \tilde{\gamma}_i = h^\mu_i\gamma_\mu \quad , \tag{2.27}$$

The spinor covariant derivatives are $\nabla_\mu = \partial_\mu + \Gamma_\mu$ where Γ_μ are the connections. By choosing the appropriate basis of one-forms (such that $u_\mu dx^\mu$ is one of the elements of this basis) it is not difficult to show that

$$\xi^\mu \Gamma_\mu = \frac{1}{4}\tilde{\gamma}^t \tilde{\gamma}^i B_{,i} = \tilde{\Gamma}_t \quad , \tag{2.28}$$

where $\tilde{\Gamma}_t$ is the time-component of the spinor connection on $\tilde{\mathcal{M}}$. The Dirac operator takes the form

$$\gamma^\mu \nabla_\mu = \tilde{\gamma}^\mu (\tilde{\nabla}_\mu - a_\mu \partial_t) = \tilde{\gamma}^\mu D_\mu \quad , \tag{2.29}$$

where $\tilde{\nabla}_\mu = \partial_\mu + \tilde{\Gamma}_\mu$ are the spinor covariant derivatives on $\tilde{\mathcal{M}}$. The corresponding equation and single-particle Hamiltonian for fiducial spin 1/2 fields are

$$(\tilde{\gamma}^\mu (\tilde{\nabla}_\mu + i\lambda a_\mu) + m)\psi^{(\lambda)} = 0 \quad , \tag{2.30}$$

$$H(\lambda) = i\tilde{\gamma}_t (\tilde{\gamma}^i (\tilde{\nabla}_i + i\lambda a_i) + m) \quad . \tag{2.31}$$

The solution to (2.30) of the form $\psi^{(\lambda)}_\omega(t, x^i) = e^{-i\omega t}\psi^{(\lambda)}_\omega(x^i)$ solves (2.24) at $\lambda = \omega$. We see, therefore, that the KK method is universal for scalar and spinor fields in a sense that it does not depend on field equations. The fiducial background $\tilde{\mathcal{M}}$ and the gauge field a_μ are determined only by the Killing vector and by geometry of $\mathcal{M}$.

2.4 Conformal transformation to zero acceleration space-time

For practical purposes it is convenient to change representation of the single-particle Hamiltonians as [9],[11]

$$\bar{H}(\lambda) = e^{-\frac{D-k}{2}\sigma} H(\lambda) e^{\frac{D-k}{2}\sigma} \quad , \tag{2.32}$$

$$e^{-2\sigma} = -\xi^2 = B \quad , \tag{2.33}$$

where $k = 2$ for scalars and $k = 1$ for spinors. $\bar{H}^2(\lambda)$ are second order differential operators of the standard form

$$\bar{H}^2(\lambda) = -\bar{h}^{ij}(\bar{\nabla}_i + i\lambda a_i)(\bar{\nabla}_j + i\lambda a_j) + \bar{V}(\lambda) \quad . \tag{2.34}$$

Connections $\bar{\nabla}_i$ correspond to fields on a $D-1$ space $\bar{\mathcal{B}}$ conformally related to $\mathcal{B}$

$$d\bar{l}^2 = \bar{h}_{ij} dx^i dx^j = e^{2\sigma} dl^2 \quad . \tag{2.35}$$

For a scalar field described by (2.17) the "potential term" in (2.34) is

$$\bar{V}(\lambda) = \bar{V} = B\left[V + \frac{D-2}{2}(\nabla^\mu w_\mu - \frac{D-2}{2} w^\mu w_\mu)\right] \quad , \tag{2.36}$$

where w_μ is acceleration (2.4). For spinors

$$\bar{V}(\lambda) = \frac{1}{4}\bar{R} + B(m^2 + m\gamma^\mu w_\mu) - i\sqrt{B}\lambda\gamma^\mu\gamma^\nu A_{\mu\nu} \quad , \tag{2.37}$$

where m is the mass of the field, $\bar{R}$ is the scalar curvature of $\bar{\mathcal{B}}$ and $A_{\mu\nu}$ is the rotation tensor (2.5). The relation between $\bar{R}$ and the scalar curvature R of the physical space-time $\mathcal{M}$ is (see Appendix A)

$$\bar{R} = B[R + (D-1)(2\nabla^\mu w_\mu - (D-2)w^\mu w_\mu) - A^{\mu\nu}A_{\mu\nu}] \quad . \tag{2.38}$$

It is worth pointing out that one can derive (2.34) by a conformal transformation in the initial equations on the physical space-time. The physical metric $g_{\mu\nu}$ changes to $\bar{g}_{\mu\nu} = g_{\mu\nu}/B$ and the Killing vector on the rescaled space has the unit norm, $\xi^2 = -1$. Thus, Killing observers in space-time $\bar{g}_{\mu\nu}$ have a non-vanishing angular velocity but zero acceleration.

3 The density of energy levels

3.1 Definition

In what follows we will be dealing with problems where spectrum of energies is continuous and show how the KK method can be used in this case. Such problems appear in a number of important physical situations like quantum theory around rotating black holes. It also turns out that computations in case of continuous spectrum are simplified. A continuous spectrum is characterized by the density of levels which in non-relativistic quantum mechanics is defined as

$$\frac{dn(E)}{dE} = \int d^3x \sum_l j_0(\Psi_{E,l}) \quad . \tag{3.1}$$

Here $\Psi_{E,l}$ is a complete set of eigen-functions of the energy operator with the same eigen-value E. In (3.1), $\sum_l j_0(\Psi_{E,l})$ is the spectral density of states or the total spectral measure and j_0 is the time component of the "current"

$$j_0(\Psi) = \Psi^*\Psi \quad , \quad j_i(\Psi) = -\frac{i}{2m}\left(\Psi^*\partial_i\Psi - \partial_i\Psi^*\Psi\right) \quad , \tag{3.2}$$

where m is the mass of the particle. The set of $\Psi_{E,l}$ is normalized by using the delta-function $\delta(E-E')$ and, strictly speaking, the integral in r.h.s. of (3.1) is divergent. To avoid the divergence one has to work with a regularized density obtained by restricting the integration in (3.1) to some compact region.

Similarly, in a relativistic quantum field theory one considers a regularized density of energy levels of single-particle excitations $\Psi_{\omega,l}$. This quantity is defined by covariant generalization of (3.1) as

$$\frac{dn(\omega)}{d\omega} = \int_\Sigma d\Sigma^\mu \sum_l j_\mu(\Psi_{\omega,l}) \quad , \tag{3.3}$$

where Σ is a space-like Cauchy hypersurface and $d\Sigma^\mu$ is its volume element. The regularization means that Σ is restricted by a region where integral (3.3) converges. The current

j_μ is determined by field equations and is divergence free, $\nabla^\mu j_\mu = 0$. This property guarantees independence of (3.3) on the choice on Σ. For free scalar and spinor fields ($\Psi = \phi$ or ψ)

$$j_\mu(\phi_1, \phi_2) = -i(\phi_1^* \partial_\mu \phi_2 - \partial_\mu \phi_1^* \phi_2) \quad , \tag{3.4}$$

$$j_\mu(\psi_1, \psi_2) = \bar{\psi}_1 \gamma_\mu \psi_2 \quad , \tag{3.5}$$

where $\bar{\psi} = \psi^+ \tilde{\gamma}_t / \sqrt{B}$ is the Dirac conjugate spinor. The divergence of j_μ is zero if ϕ and ψ obey equations (2.17) and (2.24), respectively . The density (3.3) is obtained from (3.4), (3.5) when one takes $\phi_1 = \phi_2 = \phi_{\omega,l}$ and $\psi_1 = \psi_2 = \psi_{\omega,l}$. The eigen modes are assumed to be normalized with respect to the products

$$< \Psi_1, \Psi_2 > \equiv \int_\Sigma d\Sigma^\mu j_\mu(\Psi_1, \Psi_2) \quad . \tag{3.6}$$

In case of scalar fields (3.6) is the standard Klein-Gordon product.

In the KK method the single-particle modes are obtained as solutions to a fiducial problem. The fiducial fields obey equations (2.22) and (2.30) which dictate a different form of the corresponding vector currents and products, namely,

$$\tilde{j}_\mu(\phi_1, \phi_2) = -i(\phi_1^* (\partial_\mu + i\lambda a_\mu) \phi_2 - (\partial_\mu - i\lambda a_\mu) \phi_1^* \phi_2) \quad , \tag{3.7}$$

$$\tilde{j}_\mu(\psi_1, \psi_2) = \bar{\psi}_1 \tilde{\gamma}_\mu \psi_2 \quad , \tag{3.8}$$

$$(\Psi_1, \Psi_2) \equiv \int_{\tilde{\Sigma}} d\tilde{\Sigma}^\mu \tilde{j}_\mu(\Psi_1, \Psi_2) \quad . \tag{3.9}$$

In general, (3.6) and (3.9) are not equivalent and relation between physical and fiducial modes requires an additional study.

3.2 Scalar fields

To find out this relation we assume that the Killing frame has zero acceleration and put $\xi^2 = -1$. This simplifies computations without loss of generality. One can always reduce a general problem to this case by a conformal rescaling, see Section 2.4. Thus, on a constant-time hypersurface Σ one has for (3.6), (3.9)

$$< \phi_\omega, \phi_\sigma > = \int_\Sigma \sqrt{h} d^{D-1}x \left[(\omega + \sigma) \phi_\omega^* \phi_\sigma + i \phi_\omega^* a^i (\nabla_i + i\sigma a_i) \phi_\sigma - i(\nabla_i - i\omega a_i) \phi_\omega^* a^i \phi_\sigma \right] \quad , \tag{3.10}$$

$$(\phi_\omega^{(\lambda)}, \phi_\sigma^{(\lambda)}) = \int_\Sigma \sqrt{h} d^{D-1}x (\omega + \sigma) (\phi_\omega^{(\lambda)})^* \phi_\sigma^\lambda \quad , \tag{3.11}$$

where $h = \det h_{ij}$, and h_{ij} is the metric induced on Σ. (Indexes i, j are raised with the help of h^{ij}.) In problems with a continuous spectrum Σ has an infinite volume and (3.10), (3.11) are to be interpreted in the sense of distributions. We denote C_∞ the "asymptotic infinity" of Σ where integrals (3.10), (3.11) may diverge. C_∞ may be a true infinity of the space-time, as in case of fields around a rotating star. However, in general, it is a region where the coordinate system associated with the given reference frame is singular.

For instance, C_∞ may be a black hole horizon, or a surface where rotation of the Killing frame approaches the speed of light.

The behaviour of fields at C_∞ can be used for normalization. This standard procedure is based on the fact that inner products are reduced to surface integrals over C_∞. Let C_r be a boundary of some finite region Σ_r inside Σ such that at $r \to \infty$ Σ_r expands to Σ and C_r coincides with C_∞. Then, as is shown in Appendix B, at $\omega \neq \sigma$

$$(\phi_\omega^{(\lambda)}, \phi_\sigma^{(\lambda)}) = \frac{1}{(\omega-\sigma)} \lim_{r\to\infty} \int_{C_r} d\sigma^i \left(\nabla_i (\phi_\omega^{(\lambda)})^* \phi_\sigma^{(\lambda)} - (\phi_\omega^{(\lambda)})^* \nabla_i \phi_\sigma^{(\lambda)} - 2i\lambda a_i (\phi_\omega^{(\lambda)})^* \phi_\sigma^{(\lambda)}\right) \quad , \tag{3.12}$$

$$< \phi_\omega, \phi_\sigma >= \frac{1}{(\omega-\sigma)} \lim_{r\to\infty} \int_{C_r} d\sigma^i \left(\nabla_i \phi_\omega^* \phi_\sigma - \phi_\omega^* \nabla_i \phi_\sigma - i(\omega+\sigma) a_i \phi_\omega^* \phi_\sigma\right) \quad . \tag{3.13}$$

Equations (3.12) and (3.13) are to be interpreted in the sense of distributions. Suppose now that $\phi_\omega^{(\lambda)}$ admit the following normalization

$$(\phi_{\omega,k}^{(\lambda)}, \phi_{\sigma,l}^{(\lambda)}) = \delta_{lk} \delta(\omega-\sigma) \quad , \tag{3.14}$$

where indexes l, k correspond to additional degeneracy of the wave-functions. It is clear that if (3.14) holds, the modes $\phi_\omega = \phi_\omega^{(\omega)}$ have correct normalization with respect to the Klein-Gordon product,

$$< \phi_{\omega,k}, \phi_{\sigma,l} >= \delta_{lk} \delta(\omega-\sigma) \quad . \tag{3.15}$$

This follows from the fact that at $\lambda = \omega$ (3.13) is obtained from (3.12) in the limit $\sigma \to \omega$. It is for this reason identification ϕ_ω with $\phi_\omega^{(\omega)}$ is justified.

Let us consider now the regularized density of levels (3.3) of single-particle states and show how it is related to the heat kernel of the operator $H^2(\lambda)$. Consider two integrals in the region Σ_r

$$\frac{dn(\omega)}{d\omega} = \int_{\Sigma_r} \sqrt{h} d^{D-1}x \sum_k \left[2\omega |\phi_{\omega,k}|^2 + i\phi_{\omega,k}^* a^i (\nabla_i + i\omega a_i) \phi_{\omega,k} - i(\nabla_i - i\omega a_i) \phi_{\omega,k}^* a^i \phi_{\omega,k}\right] \quad , \tag{3.16}$$

$$\frac{dn^{(\lambda)}(\omega)}{d\omega} = \int_{\Sigma_r} d\Sigma^\mu \sum_k \tilde{j}_\mu (\phi_{\omega,k}^{(\lambda)}) = \int_{\Sigma_r} \sqrt{h} d^{D-1}x \sum_k 2\omega |\phi_{\omega,k}^{(\lambda)}|^2 \quad . \tag{3.17}$$

Quantity (3.17) is the regularized spectral density of $H^2(\lambda)$. Let us define also an auxiliary quantity

$$\frac{d\tilde{n}^{(\lambda)}(\omega)}{d\omega} = \frac{dn^{(\lambda)}(\omega)}{d\omega} - \frac{1}{4\lambda} \sum_k \left[(\phi_{\omega,k}^{(\lambda)}, \partial_\lambda H^2(\lambda) \phi_{\omega,k}^{(\lambda)}) + (\phi_{\omega,k}^{(\lambda)}, \partial_\lambda H^2(\lambda) \phi_{\omega,k}^{(\lambda)})^*\right] \quad , \tag{3.18}$$

where according with (2.34), (2.36),

$$\partial_\lambda H^2(\lambda) = -2ia^i (\nabla_i + i\lambda a_i) - \nabla_i a^i \quad . \tag{3.19}$$

As follows from (3.16), (3.19),

$$\frac{dn(\omega)}{d\omega} = \frac{d\tilde{n}^{(\omega)}(\omega)}{d\omega} \quad . \tag{3.20}$$

Consider now the spectral representation for the heat kernel of $H^2(\lambda)$

$$\mathrm{Tr}e^{-tH^2(\lambda)}=\int_\mu^\infty \frac{dn^{(\lambda)}(\omega)}{d\omega}e^{-t\omega^2}d\omega \quad , \tag{3.21}$$

where integration in the trace is restricted by Σ_r . The parameter μ ($\mu>0$) is the mass gap of $H^2(\lambda)$ and we assume that there are no bound states in the spectrum. We also assume that μ does not depend on λ, which is true in a number of physical problems. The spectral density can be written symbolically as

$$\frac{dn^{(\lambda)}(\omega)}{d\omega}=2\omega\mathrm{Tr}[\delta(H^2(\lambda)-\omega^2)] \quad . \tag{3.22}$$

One can also define the integral

$$\int_\mu^\infty \frac{d\tilde{n}^{(\lambda)}(\omega)}{d\omega}e^{-t\omega^2}d\omega=\mathrm{Tr}\left[\left(1-\frac{1}{2\lambda}\partial_\lambda H^2(\lambda)\right)e^{-tH^2(\lambda)}\right] \quad , \tag{3.23}$$

where the right hand side is the consequence of (3.18). Because the trace does not depend on the choice of the basis and, hence, on λ one can write (3.23) as

$$\int_\mu^\infty \frac{d\tilde{n}^{(\lambda)}(\omega)}{d\omega}e^{-t\omega^2}d\omega=\left(1+\frac{1}{2\lambda t}\partial_\lambda\right)\mathrm{Tr}e^{-tH^2(\lambda)} \quad . \tag{3.24}$$

This formula is our key relation which together with (3.20) enables us to compute the physical density of levels $dn/d\omega$ by using powerful heat kernel techniques. From (3.23) one can also derive a formal expression for $dn/d\omega$

$$\begin{aligned}\frac{dn(\omega)}{d\omega}&=2\omega\mathrm{Tr}\left[\left(1-\frac{1}{2\lambda}\partial_\lambda H^2(\lambda)\right)\delta(H^2(\lambda)-\omega^2)\right]_{\lambda=\omega}\\ &=2\omega\mathrm{Tr}\left[\delta(H^2(\lambda)-\omega^2)-\frac{1}{2\lambda}\partial_\lambda\theta(H^2(\lambda)-\omega^2)\right]_{\lambda=\omega} \quad , \end{aligned}\tag{3.25}$$

where $\theta(x)$ is the step function, and $\theta'(x)=\delta(x)$. The above results concern systems with continuous spectra. Some similar relations for discrete spectra are discussed in Appendix B.

3.3 Spinor fields

All results established for scalars can be extended to spin 1/2 fields. We again work on a zero-acceleration space-time. Then, according to (3.5), (3.8) the products of spinor functions are

$$<\psi_1,\psi_2>=\int_\Sigma\sqrt{h}dx^{D-1}\bar{\psi}_1\gamma^t\psi_2 \quad , \quad (\psi_1,\psi_2)=\int_\Sigma\sqrt{h}dx^{D-1}\bar{\psi}_1\tilde{\gamma}^t\psi_2 \quad . \tag{3.26}$$

These expressions are different because $\gamma^t=\tilde{\gamma}^t-a^i\tilde{\gamma}_i$. They are reduced to the surface integrals (see Appendix B)

$$(\psi_\omega^{(\lambda)},\psi_\sigma^{(\lambda)})=\frac{i}{(\omega-\sigma)}\lim_{r\to\infty}\int_{C_r}d\sigma^i(\psi_\omega^{(\lambda)})^+\tilde{\gamma}_i\psi_\sigma^{(\lambda)} \quad , \tag{3.27}$$

$$< \psi_\omega, \psi_\sigma >= \frac{i}{(\omega-\sigma)} \lim_{r\to\infty} \int_{C_r} d\sigma^i \psi_\omega^+ \tilde{\gamma}_i \psi_\sigma \quad . \tag{3.28}$$

Suppose that $\psi_\omega^{(\lambda)}$ are a set of modes properly normalized in the sense of distributions, see (3.14). Then by comparing (3.27), (3.28) and using the same arguments as for scalar fields we conclude that modes $\psi_\omega = \psi_\omega^{(\omega)}$ are normalized by (3.15). Let us introduce

$$\frac{d\tilde{n}^{(\lambda)}}{d\omega} = \frac{dn^{(\lambda)}}{d\omega} - \frac{\omega}{\lambda} \sum_k (\psi_{\omega,k}^{(\lambda)}, \partial_\lambda H(\lambda) \psi_{\omega,k}^{(\lambda)}) \quad , \tag{3.29}$$

$$\partial_\lambda H(\lambda) = -\tilde{\gamma}_t a^i \tilde{\gamma}_i \tag{3.30}$$

where $H(\lambda)$ is the spinor Hamiltonian (2.31) and

$$\frac{dn^{(\lambda)}}{d\omega} = \sum_k (\psi_{\omega,k}^{(\lambda)}, \psi_{\omega,k}^{(\lambda)}) \tag{3.31}$$

is the spectral density of $H(\lambda)$. Then, as follows from (3.26), (3.29)–(3.31), the density of levels of physical states is

$$\frac{dn}{d\omega} = \sum_k < \psi_{\omega,k}, \psi_{\omega,k} >= \left. \frac{d\tilde{n}^{(\lambda)}}{d\omega} \right|_{\lambda=\omega} \quad . \tag{3.32}$$

Finally, from (3.29), (3.32) one gets for spinor density $dn/d\omega$ formula (3.24).

4 High-frequency asymptotics

Formula (3.24) makes it possible to use heat kernel techniques in stationary backgrounds and find $dn/d\omega$ is one important limit, namely, in the limit of high frequencies ω. The integral in (3.24) is determined by $\omega \simeq t^{-1}$ and this limit corresponds to the asymptotic form of the heat kernel at small values of t

$$\mathrm{Tr} e^{-t\bar{H}^2(\lambda)} \simeq \frac{1}{(4\pi t)^{(D-1)/2}} \sum_{n=0}^{\infty} \left[a_n(\lambda) t^n + b_n(\lambda) t^{n+\frac{1}{2}} \right] \quad , \tag{4.1}$$

where a_n and b_n are the standard heat kernel coefficients, $n = 0, 1, 2, \ldots$. On manifolds without boundaries $b_n = 0$. Coefficients $a_n(\lambda)$ and $b_n(\lambda)$ are even functions of λ because the fiducial theory is $U(1)$ invariant and the heat coefficients are even functions of charges. The gauge invariance also guarantees that the coefficients are polynomials in powers of the Maxwell stress tensor and its derivatives. In our case the role of the gauge field is played by the vector $a_i dx^i$, and hence the corresponding Maxwell tensor is related to the rotation. In general,

$$a_n(\lambda) = \sum_{m=0}^{[n/2]} \lambda^{2m} a_{2m,n} \quad , \quad b_n(\lambda) = \sum_{m=0}^{[n/2]} \lambda^{2m} b_{2m,n} \quad , \tag{4.2}$$

where $a_{2m,n}$ do not depend on λ. The highest power of λ in (4.2) can be determined by analyzing dimensionalities. Coefficients a_0 and a_1 in (4.1) do not depend on λ.

The density of levels at high frequencies can be found from (4.1) by using (3.24). In what follows, we assume that the mass gap of the operator can be neglected. In this case one can use the inverse Laplace transform in (3.24) and simplify computations. It should be noted, however, that for operators with zero gap one has to take into account the presence of infrared singularities which come out in (3.24) at small ω. One of the possibilities to avoid this problem is to use the dimensional regularization and formally consider D as a complex parameter. It is instructive first to obtain the asymptotics for the fiducial spectral density

$$\frac{dn^{(\lambda)}}{d\omega} \simeq \frac{2\omega^{D-2}}{(4\pi)^{(D-1)/2}} \sum_{n=0}^{\infty} \left[\frac{a_n(\lambda)}{\Gamma\left(\frac{D-1}{2}-n\right)} \omega^{-2n} + \frac{b_n(\lambda)}{\Gamma\left(\frac{D-2}{2}-n\right)} \omega^{-(2n+1)} \right] \quad . \tag{4.3}$$

One can easily verify that for complex D substitution of (4.3) in (3.21) results in (4.1). For $d\tilde{n}/d\omega$ relation (3.24) results in expansion of the same form

$$\frac{d\tilde{n}^{(\lambda)}}{d\omega} \simeq \frac{2\omega^{D-2}}{(4\pi)^{(D-1)/2}} \sum_{n=0}^{\infty} \left[\frac{\tilde{a}_n(\lambda)}{\Gamma\left(\frac{D-1}{2}-n\right)} \omega^{-2n} + \frac{\tilde{b}_n(\lambda)}{\Gamma\left(\frac{D-2}{2}-n\right)} \omega^{-(2n+1)} \right] \quad , \tag{4.4}$$

$$\tilde{a}_n(\lambda) = a_n(\lambda) + \frac{1}{2\lambda}\partial_\lambda a_{n+1}(\lambda) \quad , \quad \tilde{b}_n(\lambda) = b_n(\lambda) + \frac{1}{2\lambda}\partial_\lambda b_{n+1}(\lambda) \quad . \tag{4.5}$$

Finally, by taking into account (3.20), (4.2), (4.4), (4.5) one finds the asymptotics of the physical density

$$\frac{dn(\omega)}{d\omega} \simeq \frac{2\omega^{D-2}}{(4\pi)^{(D-1)/2}} \sum_{n=0}^{\infty} \left[\frac{c_n}{\Gamma\left(\frac{D-1}{2}-n\right)} \omega^{-2n} + \frac{d_n}{\Gamma\left(\frac{D-2}{2}-n\right)} \omega^{-(2n+1)} \right] \quad , \tag{4.6}$$

$$c_n = \sum_{m=n}^{2n} \frac{\Gamma\left(\frac{D-1}{2}-n\right)}{\Gamma\left(\frac{D-1}{2}-m\right)} \left(a_{2(m-n),m} + (m-n+1) a_{2(m-n)+2,m+1} \right) \quad , \tag{4.7}$$

$$d_n = \sum_{m=n}^{2n} \frac{\Gamma\left(\frac{D-1}{2}-n\right)}{\Gamma\left(\frac{D-1}{2}-m\right)} \left(b_{2(m-n),m} + (m-n+1) b_{2(m-n)+2,m+1} \right) \quad , \tag{4.8}$$

where $\Gamma(x)$ is the gamma function. It is remarkable that (4.6) is a local functional expressed in terms of the heat-kernel coefficients of some differential operators.

Some comments about (4.6) are in order.

First, if there are no boundaries ($b_n(\lambda) = 0$) one gets from (4.6) a finite result for even dimensions, although for odd D the result is formally zero. As we will see in the next Section, the proper way of dealing with the infrared problem is to keep in (4.6) D complex till the last stage of computations. Then both for even and odd D the physical quantities determined with the help of $dn/d\omega$ are finite except, possibly, a number of standard poles. As for c_n and d_n, they remain finite for all D.

Second, it is interesting to note that when expansion in (4.6) is approximated by first two terms determined by c_0 and c_1 the physical and fiducial densities coincide, $dn/d\omega \simeq dn^{(\omega)}/d\omega$. This property can be helpful in computations, see [9].

In general, the coefficients in the leading terms in (4.6) can be immediately computed by using (4.7), (4.8). The first coefficient is trivial, $c_0 = a_0$. According to (4.7),

$$c_1 = a_1 + \left(\frac{D-1}{2} - 1\right) a_{2,2} \quad , \tag{4.9}$$

$$c_2 = a_2 + \left(\frac{D-1}{2} - 2\right) a_{2,3} + \left(\frac{D-1}{2} - 2\right)\left(\frac{D-1}{2} - 3\right) a_{4,4} \quad , \tag{4.10}$$

where $a_n = a_{0,n}$. Hence, in four-dimensional space-time

$$c_1 = a_1 + \frac{1}{2} a_{2,2} \quad , \tag{4.11}$$

$$c_2 = a_2 - \frac{1}{2} a_{2,3} + \frac{3}{4} a_{4,4} \quad . \tag{4.12}$$

Term $a_{2,2}$ is determined by the gauge part of $a_2(\lambda)$, see (4.2), and in four dimensions

$$a_{2,2} = \alpha \int_{\Sigma_r} \bar{h}^{1/2} d^3x \bar{F}^{ij} \bar{F}_{ij} \quad . \tag{4.13}$$

where $\alpha = -1/12$ for scalars and $\alpha = r/6$ for spinors, r is the dimensionality of the spinor representation. Expression (4.13) can be rewritten in terms of local angular velocity (2.11) if we note that $\bar{F}_{ij} = F_{ij} = 2A_{ik}/\sqrt{B}$ and $\bar{F}_{ij}\bar{F}^{ij} = 8B\Omega^2$.

In order to compute c_2 one needs to know contribution of gauge fields in $a_3(\lambda)$ and $a_4(\lambda)$. These terms for any spin can be obtained from results of [12],[13]. For spin zero fields

$$a_{2,3} = \frac{1}{3!} \int_{\Sigma_r} \bar{h}^{1/2} d^3x \left[\frac{1}{2}\left(\bar{V} - \frac{1}{6}\bar{R}\right) \bar{F}^{ij}\bar{F}_{ij} + \frac{1}{10} \bar{\nabla}^i \bar{F}_{ji} \bar{\nabla}_k \bar{F}^{jk} \right.$$
$$\left. - \frac{1}{15} \bar{R}^{ij} \bar{F}_{ik} \bar{F}_j{}^k - \frac{1}{30} \bar{R}^{ijkl} \bar{F}_{ij} \bar{F}_{kl} \right] \quad , \tag{4.14}$$

$$a_{4,4} = \frac{1}{4!} \int_{\Sigma_r} \bar{h}^{1/2} d^3x \left[\frac{1}{12} (\bar{F}^{ij}\bar{F}_{ij})^2 + \frac{4}{15} \bar{F}_{ij}\bar{F}_{pk}\bar{F}^{ik}\bar{F}^{pj} \right] \quad , \tag{4.15}$$

where $\bar{V}$ is "potential term" (2.36) of the scalar operator $\bar{H}^2(\lambda)$, Eq. (2.34). For spinor fields

$$a_{2,3} = r a_{2,3}^{\text{scal}} - \frac{r}{3!} \int_{\Sigma_r} \bar{h}^{1/2} d^3x \left[\frac{1}{12} (\bar{R} + 12Bm^2) \bar{F}^{ij}\bar{F}_{ij} - \frac{1}{4} \bar{F}_{ij} \bar{\nabla}^2 \bar{F}^{ij} \right] \quad , \tag{4.16}$$

$$a_{4,4} = r a_{4,4}^{\text{scal}} + \frac{1}{4!} \int_{\Sigma_r} \bar{h}^{1/2} d^3x \left[\frac{1}{16} \mathrm{Tr}(\bar{\gamma}^i \bar{\gamma}^j \bar{F}_{ij})^4 - \frac{r}{2} (\bar{F}^{ij}\bar{F}_{ij})^2 \right] \quad , \tag{4.17}$$

where $a_{2,3}^{\text{scal}}$ is given by (4.14) with $\bar{V} = \frac{1}{4}\bar{R} + Bm^2$ and $a_{4,4}^{\text{scal}}$ coincides with (4.15).

Note that (4.13)–(4.17) are expressed in terms of geometrical quantities of the rescaled three-dimensional space with metric (2.35). All quantities can be also rewritten in terms of the geometry of the physical space-time, acceleration and rotation of the chosen Killing reference frame. For instance, by using (4.11), (4.13), and (A.9) one finds that for scalar fields in $D = 4$

$$c_1 = \int_{\Sigma_r} \sqrt{-g} d^3x \frac{1}{B} \left[\frac{1}{6} R - V - \frac{2}{3} \Omega^2 \right] \quad , \tag{4.18}$$

where R is the scalar curvature of the physical space-time and V is the scalar potential. For spinor fields

$$c_1 = r\int_{\Sigma_r}\sqrt{-g}d^3x\frac{1}{B}\left[-\frac{1}{12}R-\frac{1}{2}(\nabla w - w^2)+\frac{5}{6}\Omega^2 - m^2\right] \quad , \tag{4.19}$$

where Ω is given in (2.11). Thus, rotation changes coefficients starting with c_1.

5 Some applications

5.1 Vacuum energy

Asymptotics (4.6) can be used in a number of applications. As a first example, consider computation of vacuum energy E of a free quantum field on a stationary background

$$E = \int d\Sigma_\mu \xi_\nu \langle \hat{T}^{\mu\nu}\rangle_0 \quad . \tag{5.1}$$

Here $\hat{T}^{\mu\nu}$ is the stress-energy tensor of the field, ξ is a time-like Killing vector, and the integration goes over a space-like (Cauchy) hypersurface Σ. It is convenient to choose Σ as a constant time hypersurface. The quantum state is defined as a vacuum for single-particle excitations $\hat{\phi}_\omega$ with a certain energy, $\mathcal{L}_\xi \hat{\phi}_\omega = -i\omega\hat{\phi}_\omega$. For a free field which obeys equation (2.17) the straightforward calculation gives the following formal expression

$$E = \int_\mu^\infty d\omega \frac{1}{2}\omega\frac{dn}{d\omega} \quad . \tag{5.2}$$

At large ω the integral is divergent and one can use (4.6) to study the form of the divergence. In dimensional regularization the divergent part of (5.2) in four-dimensional theory is

$$E_{\text{div}} = \frac{\mu^{D-4}}{(4\pi)^{D/2}}c_2\frac{1}{D-4} \quad , \tag{5.3}$$

where c_2 is determined by (4.12), (4.13)–(4.15). It would be interesting to investigate relation of (5.3) and corresponding divergence of the vacuum energy computed by standard covariant methods.

5.2 High-temperature asymptotics

Let us consider a quantum state of fields on $\mathcal{M}$ which is viewed by a Killing observer as a thermal state at the temperature $T = (\beta\sqrt{B})^{-1}$, where $B = -\xi^2$ and β is a positive constant. Certainly, to ensure the thermal equilibrium there must exist necessary physical conditions. We assume that in systems we study these conditions are satisfied. In this case one can describe the system by a canonical ensemble and introduce the free energy

$$F[\beta] = \eta\beta^{-1}\int_\mu^\infty d\omega \frac{dn(\omega)}{d\omega}\ln(1-\eta e^{-\beta\omega}) \quad , \tag{5.4}$$

where $\eta = +1$ for bosons and $\eta = -1$ for fermions. At high temperatures the parameter β is small and the dominant contribution in (5.4) comes out from large frequencies $\omega \simeq \beta^{-1}$ where one can use the high-frequency asymptotics. By using (4.6) for $dn/d\omega$ in (5.4) and by neglecting the gap one finds

$$F(D,\beta) = F_1(D,\beta) + F_2(D,\beta) \quad , \tag{5.5}$$

$$F_1(D,\beta) = -\frac{1}{\pi^{D/2}\beta^D}\sum_{n=0}\gamma_{D,n}\Gamma\left(\frac{D-2n}{2}\right)\zeta(D-2n)c_n\left(\frac{\beta}{2}\right)^{2n} \quad , \tag{5.6}$$

$$F_2(D,\beta) = -\frac{1}{\pi^{D/2}\beta^D}\sum_{n=0}\gamma_{D-1,n}\Gamma\left(\frac{D-2n-1}{2}\right)\zeta(D-2n-1)d_n\left(\frac{\beta}{2}\right)^{2n+1} \quad . \tag{5.7}$$

Here $\zeta(x)$ is the Riemann zeta-function. The coefficient $\gamma_D = 1$ for bosons, and $\gamma_D = 1 - 2^{2n+1-D}$ for fermions. It should be noted that in case of Bose fields (5.5) includes also an additional contribution $\frac{1}{\beta}\int d\omega \ln(\beta\omega)dn/d\omega$ which appears at small ω. Function F_2 is a pure boundary part of the free energy. It follows from (5.6) and (5.7) that F_2 is related to F_1

$$F_2(D,\beta) = \frac{1}{\sqrt{4\pi}}\left. F_1(D-1,\beta)\right|_{c_n\to d_n} \tag{5.8}$$

and it is sufficient to investigate F_1 only.

First, we remind that (5.6) is obtained in dimensional regularization. When parameter D coincides with the physical dimensionality one of the terms in (5.6) has a simple pole. This pole corresponds to an infrared singularity of the theory with zero mass gap. The pole in F_1 appears at $n = D/2$, for D even and at $n = (D-1)/2$ for D odd. By taking this into account one finds for (5.6) in three dimensions

$$F_1(D=3,\beta) \simeq -\gamma_{3,0}\frac{\zeta(3)}{2\pi}\frac{c_0}{\beta^3} - \gamma_{3,1}\frac{c_1}{4\pi\beta}\left(\frac{1}{D-3} - \ln(\beta\rho)\right)$$
$$-\frac{1}{\pi^{3/2}}\sum_{n=2}\gamma_{3,n}\Gamma\left(\frac{3}{2}-n\right)\zeta(3-2n)c_n\left(\frac{\beta}{2}\right)^{2n-3} \quad , \tag{5.9}$$

where ρ is a dimensional parameter related to the regularization. As we pointed out above, the free energy is not trivial, although density of levels (4.6) used for its computations vanishes if one goes to $D = 3$. To get from (5.6) the result in four dimensions we use the identity

$$\Gamma(z/2)\zeta(z) = \pi^{z-1/2}\Gamma((1-z)/2)\zeta(1-z) \quad .$$

It gives

$$F_1(D=4,\beta) \simeq -\gamma_{4,0}\frac{\pi^2}{90}\frac{c_0}{\beta^4} - \gamma_{4,1}\frac{1}{24}\frac{c_1}{\beta^2} + \gamma_{4,2}\frac{1}{16\pi^2}\left(\frac{1}{D-4} - \ln(\beta\rho)\right)c_2$$
$$-\frac{1}{16\pi^{5/2}}\sum_{n=3}\gamma_{4,n}\Gamma\left(n-\frac{3}{2}\right)\zeta(2n-3)c_n\left(\frac{\beta}{2\pi}\right)^{2n-4} \quad , \tag{5.10}$$

When the frame does not rotates (5.10) coincides with well known high temperature expansion [1]. Special interest is leading terms in (5.10). For scalar fields

$$F_1(\beta) \simeq -\int d^3x\sqrt{-g}\left[\frac{\pi^2}{90}T^4 + \frac{1}{24}T^2\left(\frac{1}{6}R - V - \frac{2}{3}\Omega^2\right) + O(\ln T)\right] \tag{5.11}$$

where T is the local temperature. For spinor fields

$$F_1(\beta) \simeq -r\int d^3x\sqrt{-g}\left[\frac{7\pi^2}{720}T^4\right.$$

$$\left.-\frac{1}{48}T^2\left(\frac{1}{12}R + \frac{1}{2}(\nabla_\mu w^\mu - w_\mu w^\mu) - \frac{5}{6}\Omega^2 + m^2\right) + O(\ln T)\right] \quad . \tag{5.12}$$

In the both cases rotation results in a new term $\sim T^2\Omega^2$. In principle, our results enable one to compute next terms in high-temperature expansion. In particular, the logarithmic correction ($\ln T$) to (5.11) can be found explicitly with the help of (4.14)–(4.17).

5.3 Quantum fields around rotating black holes

One of the applications where asymptotics (5.11) and (5.12) can be used is studying a quantum state of fields around a rotating black hole when fields are in thermal equilibrium and rigidly rotate with black hole with the same angular velocity Ω_H. This state is analogous to the Hartle-Hawking vacuum known for Schwarzschild black holes. It was studied in [14] and recently discussed in [15],[16]. To ensure thermal equilibrium between the black hole and fields one has to surround the black hole by a reflecting mirror which has to rotate with the velocity Ω_H.

Consider a Kerr-Newman black hole with the mass M, the charge Q, and the angular momentum $J = aM$. The metric in Boyer-Lindquist coordinates is

$$ds^2 = -\left(1 - \frac{2Mr - Q^2}{\Sigma}\right)dt^2 - 2\frac{(2Mr - Q^2)a\sin^2\theta}{\Sigma}dtd\varphi$$

$$+\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{A\sin^2\theta}{\Sigma}d\varphi^2 \quad , \tag{5.13}$$

$$\Delta = r^2 - 2Mr + a^2 + Q^2 \quad , \quad \Sigma = r^2 + a^2\cos^2\theta \quad , \tag{5.14}$$

$$A = (r^2 + a^2)^2 - \Delta a^2\sin^2\theta \quad . \tag{5.15}$$

The horizon is located at

$$r = r_+ = M + \sqrt{M^2 - Q^2 - a^2} \quad . \tag{5.16}$$

The surface gravity κ and the angular velocity Ω_H for the Kerr-Newman black hole are

$$\kappa = \frac{r_+ - M}{r_+^2 + a^2} \quad , \quad \Omega_H = \frac{a}{r_+^2 + a^2} \quad . \tag{5.17}$$

Fields in thermal equilibrium with the black hole are described by a canonical ensemble in the Killing frame with the Killing vector $\xi = \partial_t + \Omega_H \partial_\varphi$. The local temperature is $T = \kappa/(2\pi\sqrt{B})$ where $B = -g_{tt} - 2\Omega_H g_{t\varphi} - \Omega_H^2 g_{\varphi\varphi}$ and $g_{\mu\nu}$ are defined in (5.13). On the horizon $B = 0$. Thus, near the horizon the local temperature is large and asymptotics (5.11), (5.12) are very good approximation for the free energy. The result can be easily found by using formula $w_\mu = \nabla_\mu \ln B/2$ for acceleration and formula (2.11) for angular velocity. The fiducial gauge potential which determines Ω has only one non-zero component $a_\varphi = -(g_{t\varphi} + \Omega_H g_{\varphi\varphi})/B$.

In principle, because (5.11) and (5.12) are functionals of an arbitrary metric these expressions can be used to extract more information about the quantum state. If the free-energy is considered as a thermal part of quantum effective action then (5.11) and (5.12) can be used to derive the stress energy tensor. Note that (5.11), (5.12) admit stationary variations $\delta g_{\mu\nu}$ of the metric ($\mathcal{L}_\xi \delta g_{\mu\nu} = 0$) when components ξ^μ of the Killing vector are held fixed. By considering such a variation of the first leading term in (5.11) one finds for scalar fields

$$< T^{\mu\nu} >_T = -\frac{2}{\sqrt{-g}} \frac{\delta F_1}{\delta g_{\mu\nu}} = \frac{\pi^2}{90} T^4 \left(g^{\mu\nu} - 4\frac{\xi^\mu \xi^\nu}{\xi^2} \right) \quad . \tag{5.18}$$

Tensor $< T^{\mu\nu} >_T$ is divergence free and traceless and it corresponds to the stress tensor of thermal radiation around a black hole. It diverges on the black hole horizon where T is infinite. It also diverges at the surface where the Killing frame rotates with the velocity of light in agreement with arguments of [14]. However in this case our results cannot be much trusted. By using (5.11) and similar variational procedure one can find corrections to (5.18) due to curvature, acceleration and rotation. We are planing to study $< T^{\mu\nu} >_T$ in a separate publication.

6 Summary and comments

The aim of our paper was to develop a computation method applicable to rotating quantum fields and to get with its help new general results. We have shown, in particular, that asymptotic form of free energy at high temperatures can be found in terms of the heat kernel coefficients of some differential operators. These operators are interpreted as one-particle Hamiltonians of a fiducial problem in external Abelian gauge field on a static background. We hope that in some cases where computations are quite involved, like rotating black holes, our method will be the helpful and effective tool. It would be an interesting problem to compare our method with covariant Euclidean formulation of finite-temperature theory.

We considered here scalar and spinor fields. Spin 1 fields require additional study to resolve a technical difficulty connected with constraints.

Our analysis was restricted by systems with continuous spectrum. An advantage of a continuous spectrum is that it is specified by the density of levels and we are able to

relate the latter to the heat kernel of fiducial Hamiltonians. The disadvantage is that one has to work with regularized quantities. In [5]–[7] high-temperature asymptotics of rotating fields were obtained on Einstein manifolds where the space is S^3. In this case operators have discrete spectra, which can be found explicitly for conformal fields. A naive calculation of our asymptotics (5.11), (5.12) on the Einstein manifold coincides with [5]–[7] in the leading term proportional to T^4. However, the next, $T^2\Omega^2$ term, does not reproduce the result of [5],[7] and the discrepancy is not in numerical coefficients. Thus, it is an open question how (5.11), (5.12) are modified by finite-size effects.

Acknowledgements: I am grateful to V.P. Frolov and D.V. Vassilevich for helpful discussions. This work is supported in part by the RFBR grant N 99-02-18146 and NATO Collaborative Linkage Grant, CLG.976417.

A Geometry in the Killing frame

We consider the metric $g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu$ in the Killing frame characterized by the four-velocity u_μ. In coordinates where $u^\mu = (1/\sqrt{B}, 0, 0, 0)$ the metric can be written in the form

$$ds^2 = -B(dt + a_i dx^i)^2 + h_{ij} dx^i dx^j \quad , \tag{A.1}$$

where $a_i = -u_i/\sqrt{B}$. Metric tensor has the following components

$$g_{\mu\nu} = \left(\begin{array}{cc} h_{ij} - B a_i a_j & -B a_i \\ -B a_j & -B \end{array} \right) \quad , \quad g^{\mu\nu} = \left(\begin{array}{cc} h^{ij} & -a^i \\ -a^j & a^2 - B^{-1} \end{array} \right) \quad , \tag{A.2}$$

where $h^{ij} h_{jk} = \delta^i_k$, $a^i = h^{ij} a_j$, $a^2 = a^i a_i$. As follows from (A.2), $\det g_{\mu\nu} = -B \det h_{ij}$. Relation (2.14) enables one to connect the Rieman tensors on $\mathcal{M}$ and $\mathcal{B}$

$$R^\lambda{}_{\mu\nu\rho}[h] = R^\gamma{}_{\alpha\beta\sigma}[g] h^\alpha_\mu h^\beta_\nu h^\sigma_\rho h^\lambda_\gamma - A^\lambda{}_\nu A_{\mu\rho} + A^\lambda{}_\rho A_{\mu\nu} - 2 A^\lambda{}_\mu A_{\nu\rho} \quad , \tag{A.3}$$

where $A_{\mu\nu}$ is the rotation tensor (2.5). Our conventions are $R^\sigma{}_{\mu\nu\lambda} = \Gamma^\sigma_{\mu\lambda,\nu} - ....$ Formula (A.3) is similar to embedding formula for the Riemann tensor of a hypersurface, see [10]. The relation between the scalar curvatures of $\mathcal{B}$ and $\mathcal{M}$, which follows from (A.3) is

$$R[h] = R[g] + 2 R_{\mu\nu}[g] u^\mu u^\nu - 3 A^{\mu\nu} A_{\mu\nu} \quad . \tag{A.4}$$

By taking into account that for the Killing field ξ_μ

$$\nabla^2 \xi_\mu = -R^\lambda_\mu[g] \xi_\lambda \quad , \tag{A.5}$$

we find with the help of (2.4), (2.5)

$$R_{\mu\nu}[g] u^\mu u^\nu = \nabla_\mu w^\mu + A^{\mu\nu} A_{\mu\nu} \quad , \tag{A.6}$$

where w_μ is the acceleration of the frame. Therefore,

$$R[h] = R[g] + 2 \nabla_\mu w^\mu - A^{\mu\nu} A_{\mu\nu} \quad . \tag{A.7}$$

In this paper we also introduced the space $\bar{\mathcal{B}}$ which is conformally related to $\mathcal{B}$

$$d\bar{l}^2 = \bar{h}_{ij} dx^i dx^j = B^{-1} h_{ij} dx^i dx^j \quad . \tag{A.8}$$

By using (A.3) one can express geometrical quantities on $\bar{\mathcal{B}}$ in terms of quantities on $\mathcal{M}$. In particular, in four dimensions $D = 4$, one has

$$\begin{aligned} R[\bar{h}] &= B \left(R[h] + 4 \tilde{\nabla}_i w^i - 2 w^i w_i \right) \\ &= B \left(R + 6 \nabla_\mu w^\mu - 6 w_\mu w^\mu - A^{\mu\nu} A_{\mu\nu} \right) \quad , \end{aligned} \tag{A.9}$$

where $\tilde{\nabla}_i$ is the connection on $\mathcal{B}$. Finally, one can find relation between $\mathcal{M}$ and fiducial space $\tilde{\mathcal{M}}$ with metric

$$d\tilde{s}^2 = -B dt^2 + h_{ij} dx^i dx^j \quad . \tag{A.10}$$

To this aim one should note that $\mathcal{B}$ can be embedded in $\tilde{\mathcal{M}}$ as a constant time hypersurface and find embedding relation analogous to (A.3) between $R^\lambda{}_{\mu\nu\rho}[h]$ and $R^\lambda{}_{\mu\nu\rho}[\tilde{g}]$. By acting in this way we easily get

$$R[\tilde{g}] = R[g] - A^{\mu\nu}A_{\mu\nu} \quad , \tag{A.11}$$

and other similar identities.

B The inner products

Here we derive relations (3.12), (3.13), (3.27), and (3.28) for inner products. For scalar functions one has

$$(\omega^2-\sigma^2)(\phi_\omega^{(\lambda)},\phi_\sigma^{(\lambda)}) = (\phi_\sigma^{(\lambda)},H^2(\lambda)\phi_\omega^{(\lambda)})^* - (\phi_\omega^{(\lambda)},H^2(\lambda)\phi_\sigma^{(\lambda)}) =$$

$$(\omega+\sigma)\lim_{r\to\infty}\int_{C_r} d\sigma^i\left(\nabla_i(\phi_\omega^{(\lambda)})^*\phi_\sigma^{(\lambda)} - (\phi_\omega^{(\lambda)})^*\nabla_i\phi_\sigma^{(\lambda)} - 2i\lambda a_i(\phi_\omega^{(\lambda)})^*\phi_\sigma^{(\lambda)}\right) \quad . \tag{B.1}$$

This gives (3.12). Note that (B.1) also holds when Σ has additional boundaries other than C_∞ provided if fields obey Dirichlet conditions at these boundaries. To find (3.13) we begin with relation

$$<\phi_\omega,\phi_\sigma> = (\phi_\omega,\phi_\sigma) + \int_{\Sigma_r}\sqrt{h}d^{D-1}\left(i\phi_\omega^* a^i(\nabla_i + i\sigma a_i)\phi_\sigma - ia^i(\nabla_i - i\omega a_i)\phi_\omega^*\phi_\sigma\right) \tag{B.2}$$

and use the identity

$$H^2(\omega) - H^2(\sigma) = (\omega-\sigma)(-2ia^i\nabla_i - i\nabla_i a^i + (\omega+\sigma)a^i a_i) \quad . \tag{B.3}$$

This enables one to rewrite (B.2) as

$$<\phi_\omega,\phi_\sigma> = (\phi_\omega,\phi_\sigma)$$

$$+\frac{1}{2(\sigma^2-\omega^2)}\left[(\phi_\sigma,(H^2(\omega)-H^2(\sigma))\phi_\omega)^* + (\phi_\omega,(H^2(\omega)-H^2(\sigma))\phi_\sigma)\right] \quad . \tag{B.4}$$

The right hand side of (B.4) is a pure surface term over C_r which coincides with the right hand side of equation (3.13).

To prove (3.27) for spinor modes it is sufficient to see that

$$(\psi_\omega^{(\lambda)},\psi_\sigma^{(\lambda)}) = \frac{1}{(\omega-\sigma)}\left[(\psi_\sigma^{(\lambda)},H(\lambda)\psi_\omega^{(\lambda)})^* - (\psi_\omega^{(\lambda)},H(\lambda)\psi_\sigma^{(\lambda)})\right] \tag{B.5}$$

and use the fact that $\bar{\psi} = \psi^+\tilde{\gamma}_t$ on zero-acceleration space-time, see Section 2.4. To prove (3.28) we first note that for spinor Hamiltonians

$$H(\omega) - H(\lambda) = (\sigma-\omega)\tilde{\gamma}^t\tilde{\gamma}^i a_i \quad , \tag{B.6}$$

hence

$$<\psi_\omega,\psi_\sigma> = (\psi_\omega,\psi_\sigma) + \frac{1}{(\sigma-\omega)}\int_{\Sigma_r}\sqrt{h}d^{D-1}x\psi_\omega^+(H(\omega)-H(\sigma))\psi_\sigma \quad . \tag{B.7}$$

The right hand side of this equation coincides with the surface term in the right hand side of (3.28).

In this Appendix we also comment on some properties of operators with discrete spectrum. Consider scalar operator $H^2(\lambda)$ which has a discrete spectrum $\omega^2(\lambda)$

$$H^2(\lambda)\phi_\omega^{(\lambda)} = \omega^2(\lambda)\phi_\omega^{(\lambda)} \quad . \tag{B.8}$$

By differentiating the both sides of (B.8) over λ one finds

$$\partial_\lambda H^2(\lambda)\phi_\omega^{(\lambda)} + H^2(\lambda)\partial_\lambda\phi_\omega^{(\lambda)} = \partial_\lambda\omega^2(\lambda)\phi_\omega^{(\lambda)} + \omega^2(\lambda)\partial_\lambda\phi_\omega^{(\lambda)} \quad , \tag{B.9}$$

$$\partial_\lambda H^2(\lambda) = -2ia^i(\nabla_i + i\lambda a_i) - \nabla^i a_i \quad . \tag{B.10}$$

The fiducial modes $\phi_\omega^{(\lambda)}$ now have a finite norm $(\phi_\omega^{(\lambda)}, \phi_\omega^{(\lambda)})$ which we denote as $N^2(\lambda, \omega)$. Let $N^2(\omega)$ be the norm of physical functions $< \phi_\omega, \phi_\omega >$. The relation between the two norms follows from definitions (3.10), (3.11)

$$N^2(\omega) = N^2(\omega, \omega) + \int_\Sigma \sqrt{h} d^{D-1}xi\phi_\omega^{(*)} \left[2a^i(\partial_i + i\omega a_i) + \nabla^i a_i\right]\phi_\omega =$$

$$= \left(1 - \frac{1}{2\lambda}\partial_\lambda\omega^2(\lambda)\right) N^2(\omega, \lambda)\Big|_{\lambda=\omega} \quad , \tag{B.11}$$

where to get the last line we used (B.9) and assumed that possible surface terms which appear under integration by parts vanish due to boundary conditions. Equation (B.11) is an analog of equation (3.18) obtained for continuous spectra.

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